Curves in Space Multivariable Calculus X. Du

- One parameter, one degree of freedom
- $\vec{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle$ in three-space
- Usually written as a vector-valued function
- **Smooth** - A parameterization is smooth on an interval if $x(t)$, $y(t)$, and $z(t)$ have continuous first derivatives on the interval except possibly at the endpoints of the interval.
- **Piecewise smooth** - A parameterization is piecewise smooth on an interval if the parameterization is smooth along subintervals of the interval.
- **Arc length**: Riemann sum with Pythagorean Theorem
  - $s = \int_{t_1}^{t_2} |\vec{r}\,'(t)|\,dt$ $\qquad s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$ in three-space
  - $\frac{ds}{dt} = |\vec{r}\,'(t)|$
  - **Arc length function**: $s(t) = \int_0^t |\vec{r}\,'(u)|\,du$
- $\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$
- $\vec{r}\,'(t) = \langle x'(t),\ y'(t),\ z'(t) \rangle$ $\qquad \vec{r}\,'(t_o)$ is tangent to $\vec{r}(t)$ at the point $\vec{r}(t_o)$
- **Unit tangent vector** $\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$
- **Unit normal vector** $\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|} = \frac{(\vec{r}\,' \times \vec{r}\,'') \times \vec{r}\,'}{|(\vec{r}\,' \times \vec{r}\,'') \times \vec{r}\,'|}$
- **Unit binormal vector** $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t) = \frac{\vec{r}\,' \times \vec{r}\,''}{|\vec{r}\,' \times \vec{r}\,''|}$
- **Curvature** $\kappa = \left|\frac{d\hat{T}}{ds}\right| = \frac{|\hat{T}'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$ $\qquad \frac{d\hat{T}}{ds} = \kappa\hat{N}$
- **Torsion** $\tau = -\hat{N} \cdot \frac{d\hat{B}}{ds} = \frac{(\vec{r}\,' \times \vec{r}\,'') \cdot r'''}{|\vec{r}\,' \times \vec{r}\,''|^2}$ $\qquad \frac{d\hat{B}}{ds} = -\tau\hat{N}$ $\qquad \frac{d\hat{N}}{ds} = -\kappa\hat{T} + \tau\hat{B}$
- $\hat{T}$, $\hat{N}$, and $\hat{B}$ make up a right-hand coordinate system.
- **Osculating plane**: contains $\hat{T}$ and $\hat{N}$. $\hat{B}$ is normal to it.
  - Plane that most closely models the behavior of the curve at a point.
  - **Osculating circle**: Lies in the osculating plane, has radius $\frac{1}{\kappa}$.
- **Normal plane**: contains $\hat{N}$ and $\hat{B}$. $\hat{T}$ is normal to it.
  - Normal to the curve at a point.
- **Rectifying plane**: contains $\hat{T}$ and $\hat{B}$. $\hat{N}$ is normal to it.
  - Curve approaches this plane and leaves this plane. Usually 'bounces off' the plane.